Announcements:

(1). HW 5 Typo Fixed.

(2). Final Practice and Midterm Practice.

(3). Consultation Hour:

12 - 13 December

10:00 - 17:00

Hint on HW 5 Q 2(b):

If we directly sub some expression to $\beta_k$:

$$\beta_k = - \frac{\langle \vec{r}_{k+1}, \vec{p}_k \rangle_A}{\langle \vec{p}_k, \vec{p}_k \rangle_A}$$

$$= - \frac{\vec{r}_{k+1}^T A \vec{p}_k}{(-\vec{r}_k - \beta_{k-1} \vec{p}_k)^T A \vec{p}_k}$$

$$= \frac{\vec{r}_{k+1}^T A \vec{p}_k}{\vec{r}_k^T A \vec{p}_k}$$

Try to rewrite $A\vec{p}_k$ into linear combinations of $\vec{r}_{k+1}$, $\vec{r}_k$.

Q1).

(a).
$$\vec{x}_0 = \vec{u}_1 + \vec{u}_2 + \cdots + \vec{u}_n$$

$g(\vec{x}) = x_\ell$ where $\ell$ is the smallest index s.t. $|x_\ell| = \|\vec{x}\|_\infty$.

For $\vec{x}$ and $a\vec{x}$, $0 \neq a \in \mathbb{C}$,

$$|x_\ell| = \|\vec{x}\|_\infty \text{ iff } |a||x_\ell| = \|a\vec{x}\|_\infty$$

$$\therefore g(a\vec{x}) = a x_\ell = a g(\vec{x}).$$

$$\vec{x}_k = \frac{A\vec{x}_{k-1}}{g(A\vec{x}_{k-1})}$$

$$= \frac{A(A\vec{x}_{k-2} / g(A\vec{x}_{k-2}))}{g(A(A\vec{x}_{k-2} / g(A\vec{x}_{k-2})))}$$

$$= \frac{A^2\vec{x}_{k-2} / g(A\vec{x}_{k-2})}{g(A^2\vec{x}_{k-2}) / g(A\vec{x}_{k-2})}$$

$$= \frac{A^2\vec{x}_{k-2}}{g(A^2\vec{x}_{k-2})} = \frac{A^k\vec{x}_0}{g(A^k\vec{x}_0)}.$$

$$A^k\vec{x}_0 = \lambda_1^k\vec{u}_1 + \lambda_2^k\vec{u}_2 + \cdots + \lambda_n^k\vec{u}_n$$

$$g(A^k \vec{x}_0) = \lambda_1^k \, g\left(\vec{u}_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k \vec{u}_2 + \cdots + \left(\frac{\lambda_n}{\lambda_1}\right)^k \vec{u}_n\right)$$

$$\frac{g(A^{k+1} \vec{x}_0)}{g(A^k \vec{x}_0)} = \lambda_1 \frac{g\left(\vec{u}_1 + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^{k+1} \vec{u}_i\right)}{g\left(\vec{u}_1 + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^{k} \vec{u}_i\right)}$$

$$\rightarrow \lambda_1$$

(b).

$$A\vec{u}_i = \lambda_i \vec{u}_i$$

$$\Rightarrow \frac{1}{\lambda_i}\vec{u}_i = A^{-1}\vec{u}_i$$

$A^{-1}$ has eigenvalues $\frac{1}{\lambda_i}$, the same eigenvectors,

and $\frac{1}{|\lambda_n|} > \frac{1}{|\lambda_{n-1}|} > \cdots > \frac{1}{|\lambda_1|} > 0$

$$\vec{x}^k = \frac{A^{-k}\vec{x}_0}{g(A^{-k}\vec{x}_0)}$$

$$g(A\vec{x}_k) = g\left(\frac{A \cdot A^{-k}\vec{x}_0}{g(A^{-k}\vec{x}_0)}\right) = \frac{g(A^{-k+1}\vec{x}_0)}{g(A^{-k}\vec{x}_0)}$$

$$A^{-k}\vec{x}_0 = \left(\frac{1}{\lambda_n}\right)^k \vec{u}_n + \cdots + \left(\frac{1}{\lambda_1}\right)^k \vec{u}_1$$

$$g(A^{-k}\vec{x}_0) = \left(\frac{1}{\lambda_n}\right)^k g\left(\vec{u}_n + \sum_{i=1}^{n-1}\left(\frac{\lambda_n}{\lambda_i}\right)^k \vec{u}_i\right)$$

$$\therefore g(A\vec{x}_k) = \frac{g(A^{-k+1}\vec{x}_0)}{g(A^{-k}\vec{x}_0)}$$

$$= \frac{\left(\frac{1}{\lambda_n}\right)^{k-1} g\left(\vec{u}_n + \sum_{i=1}^{n-1}\left(\frac{\lambda_n}{\lambda_i}\right)^{k-1}\vec{u}_i\right)}{\left(\frac{1}{\lambda_n}\right)^k g\left(\vec{u}_n + \sum_{i=1}^{n-1}\left(\frac{\lambda_n}{\lambda_i}\right)^k \vec{u}_i\right)}$$

$$\to \lambda_n.$$

Q2)

$$A^{k-1} = Q_1 R_1 \quad , \quad A^{(k-1)} = Q_2 R_2$$

$$A^{(k-1)} = Q^{(k-1)} R^{(k-1)} = Q_2 R_2$$

$$A^{(k-1)} = R^{(k-2)} Q^{(k-2)}$$

$$= Q^{(k-2)*} A^{(k-2)} Q^{(k-2)}$$

$$= \underline{Q^{(k-2)*} \cdots Q^{(0)*}} A^{(0)} \underline{Q^{(0)} \cdots Q^{(k-2)}}$$

$$Q^{(0)} \cdots \underline{Q^{(j)} R^{(j)}} \cdots R^{(0)}$$

$$= Q^{(0)} \cdots Q^{(j-1)} \underline{A^{(j)}} R^{(j-1)} \cdots R^{(0)}$$

$$= Q^{(0)} \cdots Q^{(j-1)} R^{(j-1)} Q^{(j-1)} R^{(j-1)} \cdots R^{(0)}$$

$$= Q^{(0)} \cdots Q^{(j-2)} \underline{A^{(j-1)}} Q^{(j-1)} R^{(j-1)} \cdots R^{(0)}$$

$$= Q^{(0)} \cdots Q^{(j-2)} \underline{R^{(j-2)} Q^{(j-2)}} Q^{(j-1)} R^{(j-1)} \cdots R^{(0)}$$

$$= A^{(0)} Q^{(0)} \cdots Q^{(j-1)} R^{(j-1)} \cdots R^{(0)}$$

$$= A( \cdots )$$

$$\underbrace{Q^{(0)} \cdots Q^{(k-2)}}_{Q_1} \underbrace{R^{(k-2)} \cdots R^{(0)}}_{R_1}$$

$$= \cdots$$

$$= A^{k-1}$$

$$Q_2 R_2 = A^{(k-1)} = R^{(k-2)} Q^{(k-2)}$$

$$= Q^{(k-2)*} Q^{(k-2)} R^{(k-2)} Q^{(k-2)}$$

$$= Q^{(k-2)*} A^{(k-2)} Q^{(k-2)}$$

$$= \cdots$$

$$= Q^{(k-2)*} \cdots Q^{(0)*} A^{(0)} Q^{(0)} \cdots Q^{(k-2)}$$

$$= Q_1^* A Q_1$$

$$\therefore A = Q_1 Q_2 R_2 Q_1^*$$

$$\vec{x}_k = \frac{A\vec{x}_{k-1}}{\|A\vec{x}_{k-1}\|_\infty}$$

$$= \frac{A^k \vec{x}_0}{\|A^k \vec{x}_0\|_\infty}$$

$$= \frac{Q_1 Q_2 R_2 R_1 \vec{x}_0}{\|Q_1 Q_2 R_2 R_1 \vec{x}_0\|_\infty}$$

$$A^k = A \cdot A^{k-1}$$

$$= Q_1 Q_2 R_2 Q_1^* Q_1 R_1$$

$$= Q_1 Q_2 R_2 R_1$$

Q3)

(a) $\vec{x}^{k+1} = \vec{x}^{k} - \lambda_k \vec{d}_k$

$\Rightarrow \vec{x}^{k+1} - \vec{x}^{*} = \vec{x}^{k} - \vec{x}^{*} - \lambda_k \vec{d}_k$

$\Rightarrow \vec{e}_{k+1} = \vec{e}_k - \lambda_k \vec{d}_k$

$$A^{-1}\vec{d}_k = A^{-1}(A\vec{x}^k - \vec{b})$$
$$= \vec{x}^k - A^{-1}\vec{b}$$
$$= \vec{x}^k - \vec{x}^*$$
$$= \vec{e}_k$$

(b). $\|\vec{e}_{k+1}\|_A^2 = \vec{e}_{k+1}^{\,T} A \vec{e}_{k+1}$

$$= (\vec{e}_k - \lambda_k \vec{d}_k)^T A (\vec{e}_k - \lambda_k \vec{d}_k)$$
$$= \vec{e}_k^{\,T} A \vec{e}_k - \lambda_k \vec{e}_k^{\,T} A \vec{d}_k - \lambda_k \vec{d}_k^{\,T} A \vec{e}_k$$
$$+ \lambda_k^2 \vec{d}_k^{\,T} A \vec{d}_k$$

$$= \vec{e}_k^{\,T} A \vec{e}_k - 2\lambda_k \vec{e}_k^{\,T} A \vec{d}_k + \lambda_k^2 \vec{d}_k^{\,T} A \vec{d}_k$$

$$2\lambda_k \vec{e}_k^{\,T} A \vec{d}_k = (A^{-1}\vec{d}_k)^T A \vec{d}_k$$
$$= \vec{d}_k^{\,T} \vec{d}_k$$
$$= \frac{2(\vec{d}_k^{\,T}\vec{d}_k)^2}{\vec{d}_k^{\,T} A \vec{d}_k}$$

$$\lambda_k^2 \vec{d}_k^T A \vec{d}_k = \frac{(\vec{d}_k^T \vec{d}_k)^2}{\vec{d}_k^T A \vec{d}_k}$$

$$\begin{aligned} &\vec{e}_k^T A \vec{e}_k \\ &= (A^{-1}\vec{d}_k)^T A (A^{-1}\vec{d}_k) \\ &= \vec{d}_k^T A^{-1} \vec{d}_k \end{aligned}$$

$$\|\vec{e}_{k+1}\|_A^2 = \vec{d}_k^T A^{-1} \vec{d}_k - \frac{(\vec{d}_k^T \vec{d}_k)^2}{\vec{d}_k^T A \vec{d}_k}$$

$$= \frac{d_k^T A^{-1} \vec{d}_k \cdot \vec{d}_k A \vec{d}_k - (\vec{d}_k^T \vec{d}_k)^2}{\vec{d}_k^T A \vec{d}_k}$$

$$= \frac{(d_k^T A^{-1} \vec{d}_k \cdot \vec{d}_k A \vec{d}_k)/(\vec{d}_k^T \vec{d}_k)^2 - 1}{(\vec{d}_k^T A \vec{d}_k)/(\vec{d}_k^T \vec{d}_k)^2 \cdot (\vec{d}_k^T A^{-1} \vec{d}_k)/(\vec{d}_k^T A^{-1} \vec{d}_k)}$$

$$\leq \frac{\frac{(\mu_1+\mu_n)^2}{4\mu_1\mu_n} - 1}{\frac{(\mu_1+\mu_n)^2}{4\mu_1\mu_n} \cdot \frac{1}{(\vec{d}_k^T A^{-1} \vec{d}_k)}} \quad \left( \text{as } \frac{x-1}{x} \text{ is increasing on } x > 0 \right)$$

$$= \frac{(\mu_1-\mu_n)^2}{(\mu_1+\mu_n)^2} \|\vec{e}_k\|_A^2$$

$\square$